

RELATIONS

1. Types of Relations

- 1. Empty Relation**
A relation in which no element of A is related to any other element of A, i.e., $R = \emptyset \subset A \times A$.
- 2. Universal Relation**
A relation in which each element of A is related to every element of A, i.e., $R = A \times A$.
- 3. Identity Relation**
A relation in which each element is related to itself only. $I = \{(a, a), a \in A\}$
- 4. Reflexive Relation:**
 $(a, a) \in R$, for every $a \in A$.
- 5. Symmetric Relation:**
 $(a_1, a_2) \in R$ implies that $(a_2, a_1) \in R$, for all $a_1, a_2 \in A$.
- 6. Transitive Relation:**
 $(a_1, a_2) \in R$ & $(a_2, a_3) \in R$ implies that $(a_1, a_3) \in R$, for all $a_1, a_2, a_3 \in A$.
- 7. Equivalence Relation :**
A relation R in a set A is said to be an equivalence relation if R is reflexive, symmetric & transitive.
- 8. Inverse Relation**
Inverse relation of R from A to B, denoted by R^{-1} , is a relation from B to A is defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$.
- 9. Asymmetric Relation**
 $(x, y) \in R \Rightarrow (y, x) \notin R$
- 10. Antisymmetric:** A relation is antisymmetric if:
• For all $x, y \in X$ $\{(x, y) \in R \& (y, x) \in R\} \Rightarrow x = y$
• For all $x, y \in X$ $\{(x, y) \in R \& x \neq y\} \Rightarrow (y, x) \notin R$
- 11. Irreflexive**
R is irreflexive iff $\forall a \in A, (a, a) \notin R$
- 12. Partial order relation**
R is a partial order, if R is Reflexive, Antisymmetric and Transitive.

2. EXAMPLE:

$A = \{1, 2, 3, 4\}$. Identify the properties of relations.

$$R_1 = \{(1,1), (2,2), (3,3), (2,1), (4,3), (4,1), (3,2)\}$$

$$R_2 = A \times A, R_3 = \emptyset, R_4 = \{(1,1), (2,2), (3,3), (4,4)\}$$

$$R_5 = \{(1,1), (2,2), (3,3), (4,4), (1,2), (2,1), (4,3), (3,4)\}$$

Relation	Reflexive	Symmetric	Asymmetric	Antisymmetric	Irreflexive	Transitive
R_1	✗	✗	✗	✓	✗	✗
R_2	✓	✓	✗	✗	✗	✓
R_3	✗	✓	✓	✓	✓	✓
R_4	✓	✓	✗	✓	✗	✓
R_5	✓	✓	✗	✗	✗	✓

NOTE

If $A = \{1, 2\}$, a relation $R = \{(1, 2)\}$ on A is a transitive relation. using the similar argument a relation $R = \{(x, y) : x \text{ is wife of } y\}$ is transitive, where as $R = \{(x, y) : x \text{ is father of } y\}$ is not transitive.

3. PROPERTIES

- R is not reflexive does not imply R is irreflexive. Counter example: $A = \{1, 2, 3\}, R = \{(1, 1)\}$
- R is asymmetric implies that R is irreflexive. By definition, for all $a, b \in A, (a, b) \in R$ and $(b, a) \notin R$ This implies that for all $(a, b) \in R, a \neq b$ Thus, for all $a \in A, (a, a) \notin R$ Therefore, R is irreflexive.
- R is not symmetric does not imply R is antisymmetric. Counter example: $A = \{1, 2, 3\}, R = \{(1, 2), (2, 3), (3, 2)\}$
- R is not symmetric does not imply R is asymmetric. Counter example: $A = \{1, 2, 3\}, R = \{(1, 2), (2, 2)\}$
- R is not antisymmetric does not imply R is symmetric. Counter example: $A = \{1, 2, 3\}, R = \{(1, 2), (2, 3), (3, 2)\}$
- R is reflexive implies that R is not asymmetric. By definition, for all $a \in A, (a, a) \in R$ This implies that, both (a, b) and (b, a) are in R when $a = b$. Thus, R is not asymmetric.

4. COUNTING OF RELATION

Number of relations from set A to B = 2^{mn} , where

$$|A| = m, |B| = n$$

Number of Identity relation on a set with 'n' elements = 1

Number of reflexive relation set on a set with 'n' elements = $2^{n(n-1)}$

Number of symmetric relation set on a set with 'n' elements = $2^{n(n+1)/2}$

The number of antisymmetric binary relations possible on A is $2^n \cdot 3^{(n^2-n)/2}$

The number of binary relation on A which are both symmetric and antisymmetric is 2^n .

The number of binary relation on A which are both symmetric and asymmetric is 1.

The number of binary relation which are both reflexive and antisymmetric on the set A is $3^{(n^2-n)/2}$

The number of asymmetric binary relation possible on the set A is $3^{(n^2-n)/2}$

There are at least 2^n transitive relations (lower bound) and at most $2^{n^2} - 2^{\frac{n^2-n}{2}} + 1$ (upper bound)

5. OPERATION ON RELATIONS:

$$1. R_1 - R_2 = \{(a,b) \mid (a,b) \in R_1 \text{ and } (a,b) \notin R_2\}$$

$$2. R_2 - R_1 = \{(a,b) \mid (a,b) \in R_2 \text{ and } (a,b) \notin R_1\}$$

$$3. R_1 \cup R_2 = \{(a,b) \mid (a,b) \in R_1 \text{ or } (a,b) \in R_2\}$$

$$4. R_1 \cap R_2 = \{(a,b) \mid (a,b) \in R_1 \text{ and } (a,b) \in R_2\}$$

PROPERTIES

1) If R_1 and R_2 are reflexive, and symmetric, then $R_1 \cup R_2$ is reflexive, and symmetric.

2) If R_1 is transitive and R_2 is transitive, then $R_1 \cup R_2$ need not be transitive.

counter example: Let $A = \{1,2\}$ such that $R_1 = \{(1,2)\}$ and $R_2 = \{(2,1)\}$, $R_1 \cup R_2 = \{(1,2), (2,1)\}$ and $(1,1) \notin R_1 \cup R_2$ implies that $R_1 \cup R_2$ is not transitive.

3) If R_1 and R_2 are equivalence relations, then $R_1 \cap R_2$ is an equivalence relation.

4) If R_1 and R_2 are equivalence relations on A ,

- $R_1 - R_2$ is not an equivalence relation (reflexivity fails).
- $R_1 - R_2$ is not a partial order (since $R_1 - R_2$ is not reflexive).
- $R_1 \oplus R_2 = R_1 \cup R_2 - (R_1 \cap R_2)$ is neither equivalence relation nor partial order (reflexivity fails)

5) The union of two equivalence relation on a set is not necessarily an equivalence relation on the set.

6) The inverse of an equivalence relation R is an equivalence relation.

6. COMPOSITION OF RELATIONS

Let $R_1 \subseteq A \times B$ and $R_2 \subseteq B \times C$, Composition of R_2 on R_1 , denoted as $R_1 \circ R_2$ or simply $R_1 R_2$ is

$$R_1 R_2 = \{(a,c) \mid a \in A, c \in C \wedge \exists b \in B \text{ such that } ((a,b) \in R_1, (b,c) \in R_2)\}$$

NOTE

$$R_1 (R_2 \cap R_3) \subseteq R_1 R_2 \cap R_1 R_3$$

$$R_1 (R_2 \cup R_3) = R_1 R_2 \cup R_1 R_3$$

$$R_1 \subseteq A \times B, R_2 \subseteq B \times C, R_3 \subseteq C \times D. (R_1 R_2) R_3 = R_1 (R_2 R_3)$$

$$(R_1 \circ R_2)^{-1} = R_2^{-1} \circ R_1^{-1}$$

7. EQUIVALENCE CLASS

Equivalence class of $a \in A$ is defined as $[a] = \{x \mid (x,a) \in R\}$, that is all the elements related to a under the relation R .

Example

E =Even integers, O =odd integers.

- All elements of E are related to each other and all elements of O are related to each other.
- No element of E is related to any element of O and vice-versa.
- E and O are disjoint and $Z = E \cup O$

The subset E is called the equivalence class containing zero and is denoted by $[0]$.

Properties: consider an equivalence relation R defined on a set A .

$$1. \bigcup_{a \in A} [a] = A$$

$$2. \text{For every } a, b \in A \text{ such that } a \in [b], a \neq b \text{ it follows that } [a] = [b]$$

$$3. \sum_{x \in A} |[x]| = |R|$$

$$4. \text{For any two equivalence class } [a] \text{ and } [b], \text{ either } [a] = [b] \text{ or } [a] \cap [b] = \phi$$

$$5. \text{For all } a, b \in A, \text{ if } a \in [b] \text{ then } b \in [a]$$

$$6. \text{For all } a, b, c \in A, \text{ if } a \in [b] \text{ and } b \in [c], \text{ then } a \in [c]$$

$$7. \text{For all } a \in A, [a] \neq \phi$$

Congruence modulo n given by $a \equiv b \pmod{n}$ if and only if n divides $(a - b)$.

8. BINARY OPERATIONS

Let S be a non-empty set. A function $f : S \times S \rightarrow S$ is called a binary operation on set S .

Note

Number of binary operations on a set containing n elements is n^{n^2}